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Linearization of Unsteady Transonic Flows Containing Shocks

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The problem of determining unsteady airloads on a thin, three-dimensional, planar wing oscillating with infinitesimal amplitude in a transonic flow is considered. The flow is assumed to be governed by the transonic small disturbance equation. The unsteady disturbance is taken to be a small perturbation superposed on a given steady mean flowfield. The equations governing the unsteady field, allowing for induced oscillations of any embedded shocks, are obtained. The linearization is shown to fail, locally, at the intersection of a shock with the wing surface, although the failure has little influence on the sectional characteristics of the wing.

Introduction

THE problem of determining unsteady aerodynamic loads on wings executing infinitesimal oscillation is of fundamental importance in aeroelasticity. Methods for evaluating these loads are well developed for thin profiles in subsonic and supersonic flows,[1] where, to a first approximation, the steady-state disturbance field due to thickness and mean orientation does not interact with the unsteady flow due to the oscillations. At freestream Mach numbers sufficiently close to 1, however, the steady disturbances can become large enough to influence the unsteady flow directly if the frequency is low and the aspect ratio not too small (at high frequencies or small aspect ratio the interaction can be neglected and classical methods apply[2]). Mathematical tools for predicting loads in this "strong interaction" regime have only recently been developed.

Much of the work to date has dealt with finite-difference schemes for solving various nonlinear field equations, e.g., Euler's equations,[3] the full potential equation,[4] and the transonic small disturbance equation.[5] The aeroelastician, however, is normally confronted with determining the stability of a configuration with respect to infinitesimal disturbances. For this problem a linear aerodynamic theory is desirable.

The linearization of the equations of motion in the transonic regime has been extensively discussed by Landahl,[2] who defined the region in the Mach number/frequency/aspect ratio parameter space within which steady disturbances have a first-order effect on the unsteady flow. Landahl's primary interest, however, was in solutions outside this regime, where the interaction can be neglected. Moreover, within the strong interaction region his formulation is incomplete, since it does not account for the presence of shocks, which are normally generated in the steady flow and oscillate in response to the motion of the surface. This shock motion creates a concentrated load on the surface which can represent a substantial fraction of the total unsteady loads.[6]

Two recent finite-difference studies[7,8] have considered the linearized problem within the strong interaction regime. Neither of these investigations, however, properly accounted for the displacement of the shock, and their results, therefore, must be viewed with some skepticism.

In an early paper Eckhaus[9] obtained analytical results for a two-dimensional airfoil by replacing the actual steady disturbance field by a simple normal shock discontinuity separating two uniform regions (see also Landahl's discussion in Ref. 2, Chap. 8). Eckhaus obtained (for this simplified problem) the correct jump conditions for the first harmonic disturbances at the undisturbed shock as well as the relation between the induced shock displacement and the first harmonic pressure at the shock. This approach has recently been extended by Goldstein et al.[10] to an infinite oscillating cascade. This general class of models will be discussed in a later paper by the present author.

In the present paper the linearized problem is formulated for a thin three-dimensional planar wing oscillating in a transonic flow with imbedded shocks. The jump conditions which the first harmonic potential must satisfy at the undisturbed shock location and an expression for the shock displacement (and its associated anharmonic load) in terms of the first harmonic potential are obtained. Finally, it is shown that the first harmonic pressure defined by the resulting linearized equations, will, in most cases, be singular at the shock root. This behavior could conceivably lead to numerical difficulties for a fine mesh finite-difference solution of the problem. However, the pressure singularity is integrable and does not, therefore, produce a corresponding singularity in any of the total integrated loads, such as lift.

Formulation of the Linear Problem

We now derive the equations governing the flowfield induced by small-amplitude unsteady disturbances to a steady three-dimensional transonic flow containing shocks. We will suppose that the flow is generated by a wing of root chord c fixed in a uniform stream with velocity u_∞. Distances (x,y,z) in the streamwise, spanwise, and normal directions will be measured from the leading edge of the root section in units of c. Time t will be measured in unit of the transit time c/u_∞. We assume that the velocity field is derivable from a potential, $cu_\infty(x+\phi)$, where the perturbation potential ϕ is taken to be small and governed by the transonic small disturbance equation,[2]

$$M_\infty^2(\phi_{tt}+2\phi_{xt})+(M^2-1)\phi_{xx}-\phi_{yy}-\phi_{zz}=0 \tag{1}$$

where M_∞ is the freestream Mach number and M is the local Mach number, which is related to ϕ by,

$$M^2=M_\infty^2+(\gamma+1)M_\infty^2\phi_x \tag{2}$$

To the same order, the tangency condition on the upper and lower surfaces of the wing, $z=f_\pm(x,y,t)$, is given by the mean surface approximation,

$$\phi_z=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)f_\pm(x,y,t) \quad \text{on } z=\pm 0, \quad (x,y) \text{ in } S \tag{3}$$

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(where S is the planform of the wing,) and the pressure is given by the linearized Bernoulli equation,

$$c_p = \frac{p-p_\infty}{\frac{1}{2}\rho_\infty u_\infty^2} = -2\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)\phi \tag{4}$$

Finally, if a shock is present at $x=x_s(y,z,t)$ the jump conditions which are consistent with Eq. (1) are,

$$\Delta\phi = 0 \tag{5}$$

$$\langle M^2-1\rangle = \alpha^2+\beta^2+M_\infty^2(2u_s-u_s^2) \tag{6}$$

where we have adopted the notation,

$$\Delta\psi = \psi(x_s^+)-\psi(x_s^-) \tag{7}$$

$$\langle\psi\rangle = (\psi(x_s^+)+\psi(x_s^-))/2 \tag{8}$$

For the difference and average of any quantity ψ, and where we have introduced the shock angles and velocity,

$$\alpha(y,z,t) = \frac{\partial x_s}{\partial z} = -\frac{\Delta\phi_z}{\partial\phi_x} \tag{9}$$

$$\beta(y,z,t) = \frac{\partial x_s}{\partial y} = -\frac{\Delta\phi_y}{\Delta\phi_x} \tag{10}$$

$$u_s(y,z,t) = \frac{\partial x_s}{\partial t} = -\frac{\Delta\phi_t}{\Delta\phi_x} \tag{11}$$

These relations, which are statements of the conservation of momentum across [Eq. (6)] and along [Eq. (5)] the shock, may be derived directly from the Rankine-Hugoniot conditions in the limit of weak, almost normal waves, or, more simply, by integrating Eq. (1) from $x=x_s-0$ to $x=x_s+0$ [strictly, this gives Eq. (6) if the continuity of ϕ is assumed; Eq. (5) is an integral of the conservation laws of Eqs. (9) and (10) along the shock]. We note that relations given in Eqs. (9-11) between α,β, and u_s and the potential ϕ are the derivatives of Eq. (5) with respect to y, z, and t. Finally, we observe that since the shock is, by assumption, weak, it must lie almost in a plane normal to the freestream axis, so that both the angles α and β must be small.

Equations (1-6) will be taken, for present purposes, as the "exact" statement of the problem (in addition, of course, appropriate conditions on the outer boundaries must be imposed, though they will not be stated here). We now suppose that the wing motion consists of a small-amplitude harmonic oscillation about a steady mean,

$$f_\pm(x,y,t) = F_\pm(x,y) + R_e\{f(x,y)e^{ikt}\} \tag{12}$$

where k is the reduced frequency and $|f| \ll F$. Such a motion will induce a small-amplitude, essentially harmonic, oscillation almost everywhere in the flowfield. A straightforward linearization about the steady state is not possible, however, if the steady flow contains a shock. The unsteady motion will, in general, force a small-amplitude oscillation of the shock about its mean position,

$$x_s(y,z,t) = x_{s_0}(y,z) + R_e\{x_{s_1}(y,z)e^{ikt}\};\ |x_{s_1}| \ll x_{s_0} \tag{13}$$

An observer situated between the extreme shock positions, therefore, will experience a large amplitude jump when the shock passes by. This local anharmonic effect can be accounted for by assuming an unsteady solution of the form,

$$\psi(x,y,z,t) = \psi_0(x,y,z) + R_e\{\psi_1(x,y)e^{ikt}\} - \frac{x_s-x_{s_0}}{|x_s-x_{s_0}|}H[(x-x_{s_0})(x_s-x)]Q(x,y,z,t) \tag{14}$$

where

$$Q(x,y,z,t) = \Delta_0\psi_0 + (x-x_{s_0})\Delta_0\psi_{0_x} + R_e\{e^{ikt}\Delta_0\psi_1\}$$

$$H(\xi) = \begin{cases} 0 & \xi<0 \\ 1 & \xi>0 \end{cases}$$

and where ψ represents ϕ or any of its derivatives. The subscript 0 on the operator Δ indicates that the difference is taken across the mean shock position x_{s_0}.

The first two terms in Eq. (14) represent the steady state and first harmonic response outside the interval between the mean and instantaneous shock positions. This part of the solution is discontinuous across the undisturbed shock. The final term in Eq. (14) cancels this discontinuity and transfers it to the instantaneous shock position, where [setting $x=x_s\pm$ in (14)]:

$$\Delta\psi = Q(x_s,y,z,t) = \Delta_0\psi_0 + R_e\{(\Delta_0\psi_1 + x_{s_1}\Delta_0\psi_{0_x})e^{ikt}\} \tag{15}$$

The preceding representation of the unsteady solution assumes that the potential has a regular Taylor series expansion on either side of the shock. In general this will be true everywhere except at the intersection of the shock and the wing surface, where the shock curvature and pressure gradient (on the subsonic side) are both logarithmically infinite. The effect of this singularity on the unsteady disturbance field will be discussed later.

Within the linear approximation the unsteady flow is characterized by the complex amplitudes $\phi_1(x,y,z)$ and $x_{s_1}(y,z)$. The equations determining these quantities will now be derived. We note, first, that the shock displacement is related to the potential ϕ_1 by the requirement that the total potential ϕ must be continuous at the instantaneous shock [see Eq. (5)]. Thus, setting $\psi=\phi$ in Eq. (15), we find that (since $\Delta_0\phi_0=0$),

$$x_{s_1} = -\Delta_0\phi_1/\Delta_0\phi_{0_x} \tag{16}$$

i.e., the jump in ϕ_1 across the undisturbed shock is proportional to the displacement of the shock.

The potential ϕ_1 itself satisfies the linearized form of Eq. (1),

$$[(M_0^2-1)\phi_{1x}]_x - \phi_{1yy} - \phi_{1zz} + M_\infty^2(2ik\phi_{1x} - k^2\phi_1) = 0 \tag{17}$$

where M_0 is the steady-state Mach number,

$$M_0^2 = M_\infty^2 + (\gamma+1)M_\infty^2\phi_{0x} \tag{18}$$

subject to the surface tangency constraint [from Eqs. (3, 12, and 14)]:

$$\phi_{1z} = \left(\frac{\partial}{\partial x}+ik\right)f(x,y) \quad \text{on } z=\pm 0, (x,y) \text{ in } S \tag{19}$$

Equations (17-19) (together with appropriate outer boundary conditions) completely determine ϕ_1 if no shocks are present (and, therefore, M_0 is continuous). If a shock is present in the steady flow we require, in addition, a jump condition on ϕ_1 at that point. This jump condition may be found either by linearizing the momentum balance in Eq. (6) or, much more simply, by integrating Eq. (17) across the steady-state shock, recognizing that ϕ_1 itself must be discontinuous there. Either method yields the compatibility condition,

$$(\Delta_0 M_0^2)\langle\phi_{1x}\rangle_0 = -2ikM_\infty^2\Delta_0\phi_1 - 2\alpha_0^{\frac{1}{2}}\frac{\partial}{\partial z}(\alpha_0^{\frac{1}{2}}\Delta_0\phi_1) - 2\beta_0^{\frac{1}{2}}\frac{\partial}{\partial y}(\beta_0^{\frac{1}{2}}\Delta_0\phi_1) \tag{20}$$

where $\alpha_0 = \partial x_{s0}/\partial z$, $\beta_0 = \partial x_{s0}/\partial y$ are the steady-state shock angles.

In summary, we have found that the potential ϕ_1 is the solution of the reduced wave Eq. (17), subject to the airfoil tangency condition of Eq. (19) and the shock jump condition of Eq. (20). Once ϕ_1 has been determined the shock displacement is given by Eq. (16). The total potential is then given by Eq. (14) and the pressure field by Eq. (4).

Aerodynamic Coefficients

It will be observed that, although the pressure disturbance is not everywhere harmonic (because of the moving shock), its region of anharmonicity is vanishingly small. Consequently all the global aerodynamic coefficients, such as lift or pitching moment, are harmonic in time. Let $g(x)$ be any integrable function which is single valued at $x = x_{s0}$. Then, from Eqs. (4, 14, and 16), the corresponding sectional moments of the surface pressure coefficients $c_{p\pm}(x,y,t) \equiv c_p(x,y,\pm 0,t)$ are (neglecting nonlinear terms in the unsteady amplitude)

$$\int_a^b dx g(x) c_{p\pm}(x,y,t) = 2\int_a^b dx g(x) \phi_{0x}(x,y,\pm 0)$$

$$+ 2R_e\left\{e^{ikt}\left[\int_a^b dx g(x)\left(\frac{\partial}{\partial x} + ik\right)\phi_1(x,y,\pm 0)\right.\right.$$

$$\left.\left. + (g[x_{s0}]\Delta_0\phi_1)_{\pm 0}\right]\right\} \tag{21}$$

where the last term (proportional to $\Delta_0\phi_1$) arises from the motion of the shock (and a,b are the leading and trailing edges of the y section).

In particular, the sectional lift and pitching moment (about $x=0$) can be written in the conventional form,

$$c_L(y) \equiv \int_a^b dx(c_{p-} - c_{p+}) = c_{L_0} + R_e\{c_{L_1}e^{ikt}\} \tag{22a}$$

$$c_M(y) \equiv \int_a^b dx x(c_{p-} - c_{p+}) = c_{M_0} + R_e\{c_{M_1}e^{ikt}\} \tag{22b}$$

where the first harmonic amplitudes are related to the potential ϕ_1 by [with $\hat{\phi}_1(x,y) \equiv \phi_1(x,y,+0) - \phi_1(x,y,-0)$]

$$c_{L_1}(y) = -2\hat{\phi}_1(b,y) - 2ik\int_a^b dx\hat{\phi}_1(x,y) \tag{23a}$$

$$c_{M_1}(y) = -2b\hat{\phi}_1(b,y) + 2\int_a^b dx(1 - ikx)\hat{\phi}_1(x,y) \tag{23b}$$

since $\hat{\phi}_1(a,y) = 0$. In this form (with ϕ_1 rather than ϕ_{1x} appearing under the integrals), the shock point loads do not appear explicitly.

Flow at the Shock Foot

It has been known for some time that, in steady inviscid flows, wherever a shock intersects a smoothly curved rigid surface, the fluid just downstream undergoes a rapid expansion, the pressure gradient normal to the shock and the shock curvature being infinite at the shock root.[11,12] We will show that this behavior is not significantly altered by the small disturbance approximation or motion of the bounding surface. We will show also that if such a singularity is present in the steady-state solution, the linearization given in Eq. (14) must fail near the base of the shock. This failure, however, is not severe and can be ignored in the evaluation of the unsteady aerodynamic coefficients.

We will assume, for simplicity, that the flow is two dimensional. This is not a severe restriction, since we will be concerned only with local phenomena which can always be made effectively two dimensional by a (local) Galilean transformation and rotation of the axes.

Let the shock intersect the (airfoil) surface at a point $x_{s_r}(t)$ where the prescribed normal velocity [Eq. (3)] is regular and can, therefore, be represented locally by a Taylor expansion,

$$\phi_z(x,0,t) = w_r(t) + a_r(t)(x - x_{s_r}) + \ldots \tag{24}$$

The shock, then, must be normal to the x axis at the root, i.e., from Eq. (9), $\alpha(0,t) = 0$. This implies that the velocity ϕ_x along the supersonic side of the shock must be of the form,

$$\phi_x(x_s^-,z,t) = u_{1r}(t) + a_r(t)z + \ldots \tag{25}$$

for z sufficiently small. The jump condition in Eq. (6) then determines the corresponding velocity along the subsonic sides

$$\phi_x(x_s^+,z,t) = u_{2r}(t) - a_r(t)z + \ldots \tag{26}$$

where u_{2r} is known in terms of u_1 and the shock speed $u_s = \dot{x}_s$ [see Eqs. (27) and (28)]. Transverse variations in shock speed are negligible since $\partial u_s/\partial z = \partial\alpha/\partial t = 0$ at $z = 0$. Also, we have assumed that the shock curvature is not too large; $\alpha \ll z^{1/2}$ as $z \to 0$ (this will be verified a posteriori).

Comparing Eqs. (24) and (26) we see that ϕ_{zx} along the airfoil is not equal to ϕ_{xz} along the shock. Consequently, the shock root must be a singularity for the subsonic flow behind the shock.

To analyze this singularity we consider a coordinate system fixed in the moving shock, $\xi \equiv x - x_{sr}(t)$. In this frame the relative Mach numbers on either side of the shock are

$$\bar{M}_{jr}^2 = M_\infty^2 + (\gamma+1)M_\infty^2 u_{jr}(t) + M_\infty^2(u_{sr}^2 - 2u_{sr}),\ j = 1,2 \tag{27}$$

where $\bar{M}_{1r} > 1 > \bar{M}_{2r}$. Since $\alpha(0,t) = 0$, these Mach numbers are related by the normal shock jump condition [Eq. (6)],

$$(\bar{M}_{1r}^2 - 1) + (\bar{M}_{2r}^2 - 1) = 0 \tag{28}$$

In the shock fixed frame the field Eq. (1) takes the form

$$(\bar{M}_{jr}^2 - 1)\phi_{\xi\xi} - \phi_{zz} = \ldots;\quad j=1,\ \xi<0;\ j=2,\ \xi>0 \tag{29}$$

where the omitted terms include lower-order spatial derivatives and time derivatives, both of which we expect will remain bounded in the limit $\xi \to 0$, $z \to 0$.

The flow in the supersonic region $\xi < 0$ is not influenced (locally) by the presence of the shock and we expect the solution in that region to be regular [this was assumed in writing Eq. (25)].

On the subsonic side the potential must satisfy Eq. (29) which is essentially Laplace's equation, subject to the Neumann conditions (26) and (24) along the shock and airfoil. The solution is easily found using complex variables,

$$\phi_\xi - i\frac{\phi_z}{\beta_r} = u_{2r} - \frac{iw_r}{\beta_r} + \frac{4}{\pi}\left(\frac{a_r}{\beta_r}\right)\zeta\ln\zeta + \left(\frac{b - ia_r}{\beta_r}\right)\zeta + \ldots \tag{30}$$

where

$$\beta_r \equiv (1 - \bar{M}_{2r}^2)^{1/2},\quad \zeta \equiv \xi + i\beta_r z$$

and where b is an arbitrary real constant. Along the airfoil surface, then, the axial velocity is,

$$\phi_x = \phi_\xi = u_{2r} + \frac{4}{\pi}\left(\frac{a_r}{\beta_r}\right)\xi\ln\xi + \ldots,\quad z=0,\ \xi>0 \tag{31}$$

while along the shock,

$$\phi_z = w_r - \frac{4}{\pi} a_r \beta_r z \ln z + \ldots, \quad \xi = 0+, \quad z > 0 \tag{32}$$

where we have neglected regular terms. From Eqs. (32) and (9) we see that the local shock angle is,

$$\alpha = -\left[\frac{2}{\pi}(\gamma+1) M_\infty^2 \frac{a_r}{\beta_r}\right] z \ln z + \ldots \tag{33}$$

which is small compared to $z^{1/2}$ as assumed.

We have established, then, that as long as $a_r(t) = \phi_{zx}[x_s(0),0,t] \neq 0$, both the surface velocity gradient and shock curvature are logarithmically infinite at the shock root. The local flow structure (to the order considered here) is fully determined by the shock's position, speed, and strength (i.e., x_{sr}, u_{sr}, and u_{1r}).

These results are valid for three-dimensional flows as well if we consider the coordinates x,y to be measured along the local normal and tangent to the line of intersection of the shock and wing. The pressure gradient on the wing normal to the shock and the shock curvature in the plane normal to the wing and shock are both logarithmically infinite.

We now consider the effect of these gradient singularities on the linearized unsteady problem. In general the steady shock curvature $\alpha_0'(z)$ will be logarithmically infinite as $z \to 0$. To lowest order, therefore, the jump condition in Eq. (20) is,

$$(\Delta_0 M_0^2) \langle \phi_{1x} \rangle_0 = -\frac{d\alpha_0}{dz} \Delta_0 \phi_1 + \ldots \quad \text{as} \quad z \to 0 \tag{34}$$

Since $\Delta_0 \phi_1$ is not zero at the shock root ($\Delta_0 \phi_1 = 0$ would imply that the shock remains stationary, by Eq. (16)], this implies that the first harmonic velocity ϕ_{1x} blows up logarithmically on $x = x_{s0}$ as $z \to 0$.

It can be shown, then (from an analysis essentially identical to the one used previously) that ϕ_{1x} is also singular along the airfoil surface,

$$\phi_{1x} = \left\{\frac{2}{\pi}(\gamma+1) M_\infty^2 \frac{F''}{|{\textstyle\frac{1}{2}}\Delta_0 M_0^2|^{3/2}} \Delta_0 \phi_1\right\}_{r_0} \ln|x - x_{sr}| + \ldots$$

$$\text{on } z = 0 \tag{35}$$

where the subscript r_0 indicates that the quantity in braces is evaluated at the root of the steady shock.

In fact, however, as we have already seen, the unsteady solution should be no more (and no less) singular than the steady solution. The inconsistency is created by the linearization ansatz of Eq. (14), which cannot be valid near the shock root if the steady solution is singular (and so cannot be analytically continued).

This failure, however, is purely local. The linearization in Eq. (14) remains valid everywhere except at the downstream side of the shock root. Moreover, the velocity singularity is integrable so that the potential ϕ_1 itself is well defined. Thus, the gross aerodynamic coefficients can still be obtained from Eq. (22) without any special measures to correct the velocity distribution at the shock.

If a uniformly valid representation of the surface velocity (and pressure) is required, it may be constructed using Eq. (14) away from the shock and the exact local solution, Eq. (31), at the shock, the shock displacement and speed being given by Eq. (16) and the shock strength u_{1r} by analytic continuation on the supersonic side.

Finally, it should be observed that although the nonanalytic nature of the solution is inherent in the inviscid equations of motion, it will not occur in reality. Both the surface pressure discontinuity and associated gradient singularity will always be obliterated by the strong interaction of the shock and boundary layer.

The boundary layer can have a substantial influence not only locally, by smoothing the transition through the shock, but also globally, by shifting the position of the shock. In principle, the unsteady solution could be improved by using these viscous effects while retaining the inviscid description of the unsteady perturbations.

Conclusions

The equations determining the unsteady flow generated by infinitesimal oscillations of a lifting surface in an inviscid transonic flow containing imbedded shocks have been derived. The shock excursion and total aerodynamic loads on the wing have been related to a complex potential amplitude ϕ_1 which satisfies the usual linearized transonic small disturbance equation and surface tangency condition. The field equation for ϕ_1 depends on the steady flow through the steady Mach number M_0 and is hyperbolic when $M_0 > 1$ and elliptic when $M_0 < 1$. The potential ϕ_1 is discontinuous at the discontinuities of M_0 (i.e., at the mean shock position) and must be made to satisfy a compatibility condition across such surfaces.

In practice, except for the simplest steady flows, the boundary value problem for ϕ_1 will have to be solved numerically. Although the numerical problem is relatively simple, the unsteady flow so computed can be no better than the steady flow about which the linearization was performed. In many cases, therefore, it may be desirable to replace the Mach number distribution M_0 obtained from the transonic small disturbance equation by a more realistic choice.

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